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Preservice Mathematics Teachers' Proving Skills in an Incorrect Statement: Sums of Triangular Numbers

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Abstract

The aim of this study is to examine preservice mathematics teachers' proving skills in an incorrect statement. In this way, it was tried to examine their reasoning and proving skills about the correctness of the given mathematical expression. The case study, one of the qualitative research designs, was adopted in the study. The participants of the study are 47 preservice mathematics teachers studying in the fourth grade. The data were first collected in writing with the question of "sum of triangular numbers". Afterwards, semi-structured interviews were conducted with five volunteer preservice mathematics teachers. Content analysis was used in the analysis of the written data. The findings showed that the preservice teachers did not question when expressed with "prove or show that it was true", they did not have knowledge about alternative proof methods, and they insisted on using the proof methods they were familiar with. In the light of the findings to be obtained, alternative proof methods have been tried to be presented.

Keywords: Proof, Mathematical Induction, Triangular Numbers, Preservice Mathematics Teachers, Mathematics Education

Introduction

Mathematical thinking and reasoning skills are one of the most important goals of mathematics education. In order to develop mathematical thinking and reasoning skills, environments should be provided for students to ask questions such as how and why and to seek answers to these questions. Proof plays an important role in the development of these skills. Proof is defined as showing the correctness of an assumption or statement, convincing someone about the correctness of a statement, telling what a claim means (Almeida, 2000; 2003, Hanna, 2000; Heinze & Reiss, 2004, Knuth, 2002a; 2022b; Rodd, 2000). Of course, the purpose of mathematical proof is to show the truth of a claim as well as its falsity (Lakatos, 1976). Therefore, both justification and falsification help to show whether mathematical statements are true or false (Ko & Knuth; 2009; Ko, 2010). In order to do this, our knowledge and skills about proof techniques gain importance. There are various proof techniques used in mathematics. Frequently used proof techniques can be expressed as direct proof, inductive method, proof by contraposition, proof by contradiction, and giving examples to the contrary. The method of proof by induction is used to show that a proposition defined on the set of natural numbers is true for all natural numbers, in other words, the proposition "$\forall n \in N$, p(n)" is true for the open proposition p(n) given on the set of natural numbers (Argün, Arıkan, Bulut & Halıcıoğlu, 2014).

When the literature is examined, it is seen that students have a negative attitude towards proof, they will not succeed and they are afraid of proving (eg, Almeida, 2003; de Villiers, 1990; Doruk, Kıymaz, Horzum, & Morkoyunlu, 2014; Doruk, Özdemir, & Kaplan, 2014; Gökkurt & Soylu, 2012). In addition, it is stated that students cannot prove any mathematical relationship or cannot understand a given proof (eg, Ko & Knuth, 2009; Moore, 1994) and have difficulty in creating proofs (Weber, 2001). As Doruk (2019) stated, studies on proof show that students have difficulties in interpreting mathematical definitions, understanding concepts and theorem, using mathematical language and notation, choosing the appropriate proof technique, expressing their thinking. İmamoğlu (2010) stated that first-year students in mathematics, primary and secondary mathematics education departments use inductive reasoning when creating a mathematical proof, and most of the final-year students try to use deductive methods because they need more generalization. Baker (1996), on the other hand, stated that most of the high school and university students who participated in the study focused on the practical aspect of mathematical induction rather than its conceptual dimension. In the study conducted by Yenilmez and Ev Çimen (2012), it was revealed that students had more difficulty in learning the "complex numbers and induction" subjects compared to other subjects. In the study conducted by Güler, Özdemir and Dikici (2012), preservice mathematics teachers' ability to prove by mathematical induction method

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and their views on mathematical proof were examined and the relationship between them was investigated. The findings of the study showed that preservice teachers' ability to prove with the induction method was low and there was a statistically positive and significant relationship between their views on proof and their ability to prove with the induction method.

The source of the difficulties related to proof is that most of the students do not know exactly how to prove, where to start proving and which tools to use while proving (Weber, 2001). As a matter of fact, students do not have the same knowledge in providing process, and the ways they use while proving differ (İskenderoğlu, 2010). Therefore, Sowder and Harel (1998) used the term proof scheme to describe the proofs used. Indeed, the process of making evidence and the evidence schemes used in this process are a product of the individual's thinking process and are a way of thinking that shows how students think. Sowder and Harel, while evaluating the students' solutions to mathematical problems, divided the schemas into three main titles as external proof scheme, experimental proof scheme and analytical proof scheme, and these main schemas were divided into sub-schemas. According to the External proof scheme, the student must first convince himself and then persuade others. In doing so, it uses some resources. The sources he uses for persuasion are usually external factors such as a teacher, a book, and some person he sees as an authority. At this point, students make a statement by relying on someone or something. They take what people they trust say without question. As Flores (2002) states, this is because students learn most of what they learn under the influence of the environment. External proof scheme is divided into three: authoritarian proof scheme, ritual proof scheme, and symbolic proof scheme. The empirical proof scheme usually uses examples and proof is based on examples. While learning concepts, people generally learn by examples and students try to explain what they know with examples based on this habit. Empirical proof schemes are divided into perceptual proof schemes and inductive proof schemes. In the analytical proof scheme, students use logical inferences rather than tools such as guesses, examples, and assumptions when showing why a mathematical situation is true (İskenderoğlu, 2010). While making these explanations, they obtain mathematical relations through reasoning (Flores, 2002). Analytical proof schemes are divided into two subheadings: transformational proof scheme and axiomatic proof scheme.

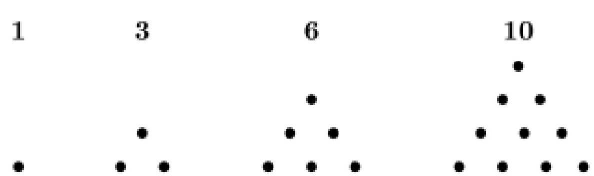


Fig. 1: Triangular numbers (Agarwal, 2021)

In the study, reasoning and proving skills were examined with the "sum of triangular numbers". Numbers that can be written as the sum of consecutive natural numbers starting from 1 are called triangular numbers. Triangular numbers are given with number patterns. The first four triangular numbers are shown in Figure 1.

Examples of visual proofs of the sum of triangular numbers are given in Figure 2a (Zerger, 1990), Figure 2b (Plaza, 2016) and Figure 2c-2d (Nelsen, 1993).

Triangular numbers are shown as T_1, T_2,... .T_n and T_n is the sum of the numbers from 1 to n. In fact, it is related to the sum of the numbers from 1 to what as an expression. The proof of the sum of numbers from 1 to n is usually done by mathematical induction method. When the literature is examined, no study has been found that examines the ability of students, prospective teachers or mathematics teachers to prove an incorrect statement. In addition, it has been observed that studies focusing on the proving skills with the mathematics induction method are limited. In this sense, it can be stated that there is a need for studies that determine the approaches for both mathematics induction method and inaccurate expression. Thanks to these studies, skills such as understanding and interpreting the mathematical expression to be proved, choosing the appropriate proof technique, creating the proof, as well as the difficulties related to proof will be examined. It is clear that the results obtained from these studies will benefit educators and the literature on proof. The aim of this study is to examine the ability of preservice elementary mathematics teachers to prove an incorrect statement. The following research questions were examined in the study:

1. What are the mathematics preservice teachers' proving skills in understanding and reasoning in proving an incorrect statement?
2. What are the difficulties experienced by preservice mathematics teachers with the method of proof by induction?

Method

Research Model

The case study method, one of the qualitative research methods, was used in the study. Gerring (2007) defines case study as the in-depth study of a single case in order to explain more cases. As a matter of fact, human behavior can be researched with a flexible and holistic approach, and in this approach, the experiences and thinking processes of the individuals participating in the research are of great importance (Yıldırım & Şimşek, 2013). Therefore, a case study is to describe and examine the process that brought about an event, to used to develop and evaluate understanding.

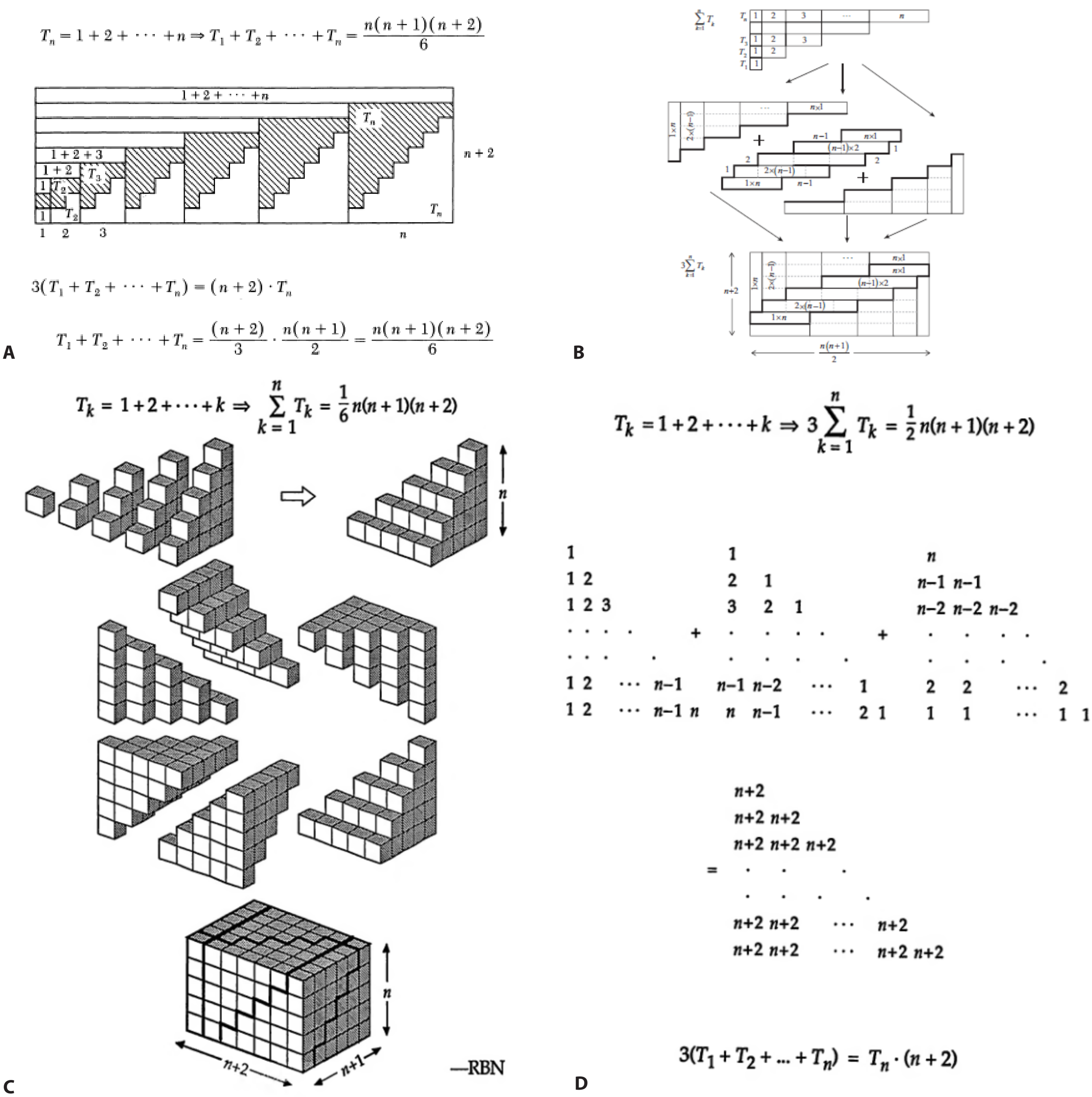


Fig. 2: **(A-D)** Sums of triangular numbers (Zerger, 1990; Plaza, 2016; Nelsen; 1993)

Participants

The participants of the study are 47 mathematics preservice teachers studying in the fourth grade of primary education mathematics teaching department of a state university located in the Central Anatolian region of Turkey. The criterion sampling method was used in the selection of the participants. The criteria mentioned here can be created by the researcher or a list of previously prepared criteria can be used (Yıldırım & Şimşek, 2013). The criterion used in the research is that, since the prospective teachers are in the fourth grade, they have knowledge about proof techniques in general, especially about induction and proof techniques, and use them in proof theorems. For the first part of the study, each participant was given a code as S1, S2, ... After examining the written answers of the participants, semi-structured interviews were conducted with five preservice teachers who volunteered and gave different answers. In this way, the maximum diversity sampling method was used. In the interviews, the preservice teachers were asked to reconsider their written answers and probing questions were asked about why they thought in this way. The interviews lasted approximately 15 minutes. Interviews were conducted through the Teams program.

Data Collection Tool

The question about the sum of the triangular numbers given below were asked to the preservice mathematics teachers.

> Triangular numbers are shown as $T_1, T_2, \ldots$ The triangular number Tn (n^{th} triangular number) is the sum of the numbers from 1 to n. The first six triangular numbers are:
>
> 1, 3 (1+2), 6 (1+2+3), 10 (1+2+3+4), 15 (1+2+3+4+5), 21 (1+2+3+ 4+5+6)
>
> Show that the sum of the triangular numbers from 1 to n is

The reason for choosing this question is to express a triangular number as the "sum of consecutive numbers" and thus to find out whether the preservice teachers associate triangular numbers with the sum of consecutive numbers and whether they question while proving. Opinions were taken from two mathematics educators while creating the question. An expert suggested that especially when giving the first six triangular numbers, it is necessary to write which numbers are the sum in parentheses. In this way, he stated that it can be understood what a triangular number is. The answers of the preservice teachers to this question were collected in writing. The collected data were analyzed by content analysis method, codes and categories were obtained. Semi-structured interviews were conducted with five teacher candidates who answered in different categories.

Data Collection

The data of the research were collected in two stages. In the first stage, the answers of the preservice mathematics teachers to the question posed were collected in writing. Necessary explanations and sufficient time were given to them. In the second stage, semi-structured interviews were conducted with five participants. The interviews were collected on the online platform through the Teams program at the appropriate time of the preservice mathematics teachers and the researcher. In this way, the effects of external factors are kept to a minimum. The researcher reflected the participant's answer on the screen and probing questions were asked about his answer. Interviews were recorded through the program. Participants were aware that they were recorded.

Data Analysis

Content analysis was used to analyze written answers from preservice teachers. Codes and categories were created according to the similarities of the answers. Code and categories were presented again to the mathematics educators who were at the stage of creating the question. In two categories, different answers were received in the categories of non-verbal proof and visual proof. It has been finalized as a common opinion. The data obtained from the interviews were presented descriptively.

Findings

The findings obtained as a result of the analysis of the answers of the preservice teachers in the question directed to the proof of the sum of triangular numbers are given in Table 1. As can be seen from Table 1, the answers are grouped under four themes.

Table 1: Findings obtained from the answers of preservice mathematics teachers

Theme	*Category*	*Subcategory*	*f*
No answer			11
Trying to show the sum of numbers from 1 to n	Mathematical Induction	not shown	1
		It is true for n=1. Let's assume it's true for n. then it is also true for n+1	3
		It is true for n=1. Let's assume it is true for n=k. then also true for n=k+1	6
		It is true for n=1. Let's assume it is true for k=n. then also true for k=n+1	2
	Gaussian method		4
Explaining triangular numbers	Proof without words-visual proof		7
	Verification		1
	Visualizing triangular numbers		7
	Other		3
Misunderstanding the given statement			2

11 preservice teachers did not answer this question. The preservice teachers who did not respond stated that they knew the steps of the method of proof by induction, but they had difficulty or even could not do it while proving an unusual statement with the induction method. Interviews with S45 and S3 who did not answer the question are given below.

S45: I don't understand the method of proof by induction at all. That's why I couldn't prove it.

R: You don't have to prove it by induction. You can try different ways.

S45: But the induction method was always used in expressions like n(n+1)/2. Let it be true for k seemed to be true within k+1. We saw it in lectures. But I can't at work.

S3: It was actually very easy. It looks like the sum of consecutive numbers. But how did he add up all of them one by one.....I guess I'm stuck with triangular numbers. Let me take another look now. Actually, I think I can.

As can be seen, although S45 thought that it should be proved by the induction method due to the expression n(n+1)/2, he stated that he had no foresight about how he could prove it. On the other hand, 23 preservice teachers who participated in the study tried to prove the sum of the numbers from 1 to n. The answers of 12 preservice teachers were evaluated under the category of "proof by induction". S4, one of these preservice teachers, did not continue her answer even though she said "*Let's prove with the inductive method*" (Figure 3).

The answer of S4 was evaluated under the category of induction instead of the category of no answer. The reason for this is that the preservice teachers who did not answer stated the induction method in the interviews. But S4 wrote that the proof can be done by induction method in the first application. The answers in the category of "Proof by induction" were evaluated in three sub-categories. "It is true for n=1. Let's assume it's true for n. then it is also true for n+1", "It is true for n=1. Let's assume it is true for n=k. then also true for n=k+1" and " It is true for n=1. Let's assume it is true for k=n. then also true for k=n+1" is given in Figure 4a, Figure 4b, and Figure 4c, respectively

As can be seen from Figure 4, they all tried to prove it by induction. In all of them, they tried to show the accuracy for n=1 in the first step. In the second step, "true for n", "true for n=k" and "true for k=n" assumptions were made. In the third step, it was tried to show that it is true for n+1, true for n=k+1, and true for k=n+1. S48, S22 and S26 are the preservice teachers who answered in this category, respectively. The data obtained from the interviews are given below.

Translation: Let's show by induction method

Fig. 3: S4' s answer

R: You reviewed your answer. Is there a part of the proof that you would change or say was wrong?

S22: No…it says prove it in the question. Such proofs are always done by induction. Looking at your steps, I already did 1, n, n+1. Every step in proof is correct.

S48: Proof by induction, which we know from lectures. But of course, the definition of triangular numbers is given in the question. We didn't do it in class.

S26: Proof of the sum of triangular numbers. I checked my answer is correct. But I thought you were multiplying those parentheses. He gave an example in the question. But the proof by induction is true.

R: I see that you are drawing a figure. You tried to express triangular numbers with dots. Do you know what a triangular number is?

S22: Yes. As can be seen in the figure, it is given in the question. The first triangular number is 1, the second is 3, the third is 6. Then it goes like 10, 15, 21. A sequence..

The Gauss method is given as an alternative proof to the proof of the sum of the numbers from 1 to n. As given in Figure 5a, four preservice teachers tried to prove with the Gauss method. Seven preservice teachers tried to demonstrate its accuracy with visual proof (Figure 5b).

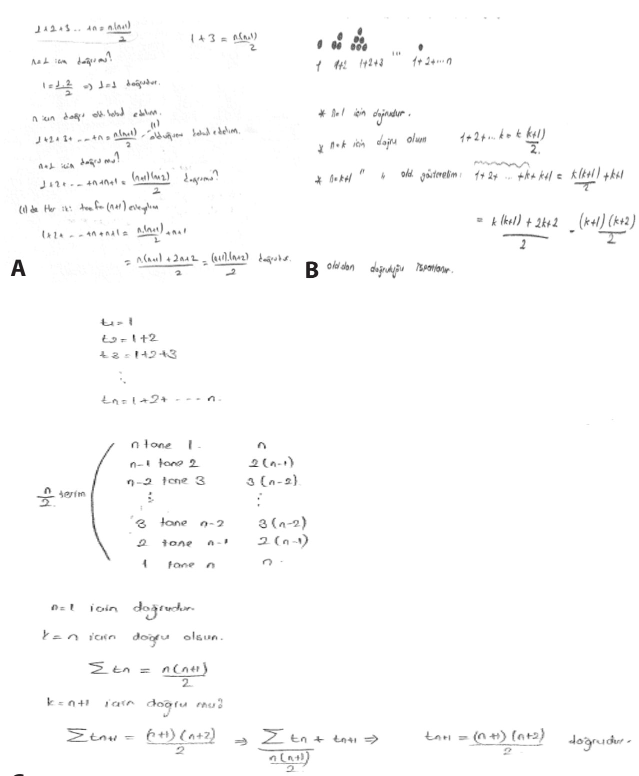
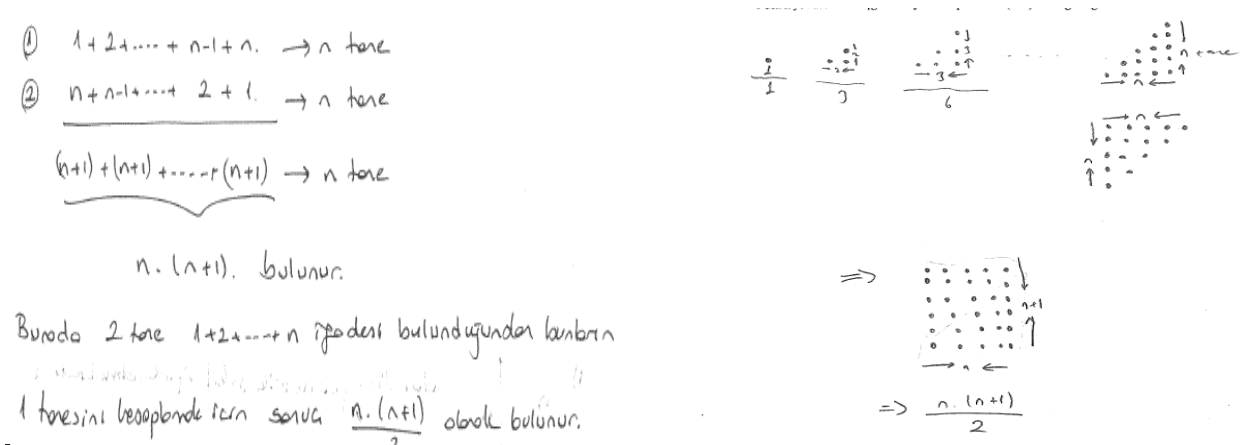


Fig. 4: Sample answers belonging to the category "Proof by induction"



A B

Fig. 5: Sample answers to the Gaussian method and nonverbal proof category

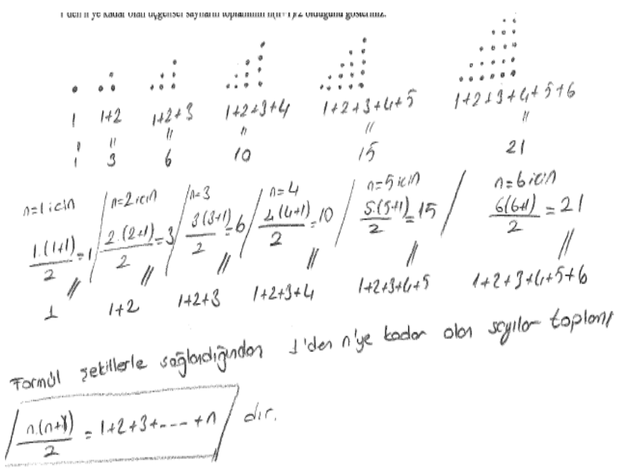


Fig. 6: An example answer in the category "Verification"

Only one preservice teacher visualized the first, second, third, fourth, fifth and sixth triangular number (Figure 6) given in the question. But he did not visualize the next triangular number. So it didn't take it one step further. So he didn't give any reasoning for the next step.He used the formula n(n+1)/2 to find each triangular number. However, instead of the expression "sum of triangular numbers from 1 to n", he still wrote "sum of numbers from 1 to n". What it actually does is find triangular numbers.

Two teacher candidates misunderstood. They replied as in Figure 7. These preservice teachers were "1, 3(1+2), 6(1+2+3), 10(1+2+3+4), 15(1+2+3+4+5), 21(1+2+3+4+5+6)" expressions are perceived as multiplication. However, before this expression, the definition of triangular numbers is given. It can be said that these preservice teachers did not pay attention to the definition given in the previous sentence.

Seven preservice teachers visualized only triangular numbers as in Figure 8a. The answers in the "Other" category (Figure 8b) are based on explaining triangular numbers

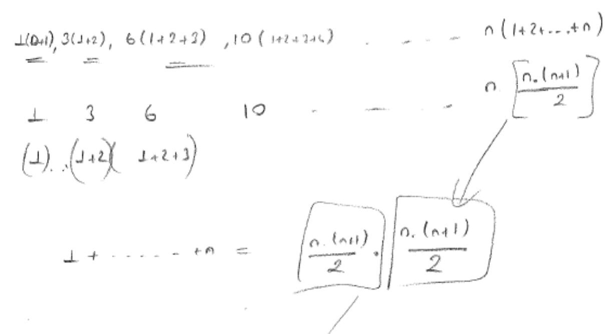
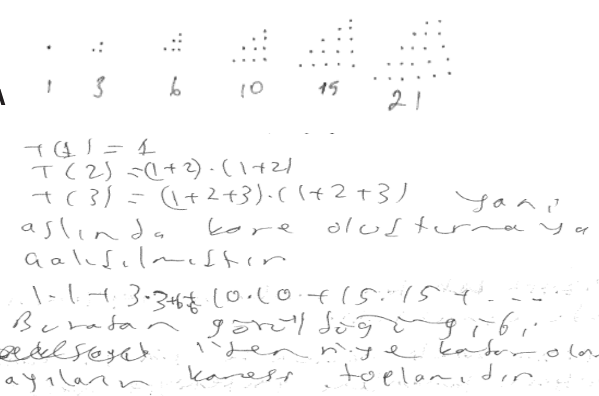


Fig. 7: An example from the category of "misunderstanding"

A



B

Fig. 8. Sample answers from the category "visualization of triangular numbers" and "other"

It is seen that the pre-service teachers in Figures 8a and 8b are trying to understand the triangular number from their answers.

Conclusion and Discussion

As a result of the research, it was revealed that none of the pre-service mathematics teachers could prove the given statement, in fact, they did not make any reasoning about the given statement. The first reason for this is the confidence that the given statement is true because of the statement "prove". For this reason, teacher candidates tried to prove it. This finding is supported by Avital and Hansen (1976). Avital and Hansen (1976) stated that exercises related to mathematical induction are expressed by the words "show that …" or "…show with mathematical induction". Therefore, they stated that the theorem whose proof was required was given by the author and that the only thing the student had to do was to prove that the statement was true. It can be supported by questions such as how theorems proved by induction are discovered using visual proofs, which is one of the alternative proof methods, and how generalizations are made.

One of the sub-dimensions of mathematical thinking is specialization. Choosing custom values when customizing is helpful to understand whether a situation or assumption is true. In other words, specialization can be done to find a counter or related example for a situation or assumption (Arslan & Yıldız, 2010). None of the pre-service teachers doubted that the statement given in the question might be wrong, and they did not take any special cases or even check their accuracy. For n=2, T_1+T_2= 1+(1+2) =1+3=4, but it can be checked that "2.3/2=3 for n=2". It can be said that the statement given in this way is not true. As Harel and Sowder (1998) stated, in the authoritative proof scheme, students trust and believe a book, a teacher, or someone they see as more knowledgeable than themselves. As a result, they do not question whether they can be wrong. Another reason is that proof is central to both mathematics and mathematics education, but it is a meaningless ritual for many students (Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002; Knuth, 2002a). Students base their proofs on rules they have memorized or learned without understanding (Harel, 2001).

Secondly, some preservice teachers focused on and answered only a part of the question. Some focused only on triangular numbers, some on the expression (n.(n+1))/2. As a result, those who focused on triangular numbers tried to visualize it. Those who focused on the expressions "sum of numbers from 1 to n" and "(n.(n+1))/2"" tried to answer with a pre-existing proof method in their minds. As a result, they showed that the sum of numbers from 1 to n is (n.(n+1))/2 by mathematical induction method. They insisted on proving what they remembered, since it was evidence they knew before. In fact, understanding what to prove is just as important as proving. The first step in problem solving is understanding the problem. In fact, the first step of making a proof can be similarly expressed as "understanding what to prove". The second stage is the "selection of the proof method" required for proof. Of course, the third step is to apply the chosen proof method.

In this study, it is seen that pre-service teachers have problems in proving with the inductive method. It shows that pre-service teachers perceive the mathematical induction method as a procedure that should be followed. It is similar to the findings of the study of Güler et. al., (2012). In summary, the results of this study are consistent with the results of studies in which pre-service mathematics teachers had difficulties in creating counterexamples and proving by induction (Doruk; 2019; Doruk & Kaplan, 2018; Güler et. al., 2012).

Recommendations

In this study, preservice elementary mathematics teachers' proving skills in case of "sum of triangular numbers" and "false statement" were examined. The data of the study were obtained from written answers and interviews of pre-service mathematics teachers. More detailed data can be obtained by collecting data using the think-aloud technique. Since it is concluded that the pre-service teachers do not understand the statement to be proved, studies can be designed to reveal the reasons for these difficulties.

Ethical Text

Within the scope of the research, ethics committee approval was obtained from the ethics committee of Sivas Cumhuriyet University with the decision dated 30.05.2022 and numbered 2021/23.

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